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A TABULATION OF THE FRESNEL INTEGRALS

By

Robert Turner
and
Anne F. Downey

March 15, 1953

Technical Report No. 173

Cruft Laboratory
Harvard University
Cambridge, Massachusetts

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Abstract

A tabulation of the Fresnel integrals

$$C(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \cos t \, \frac{dt}{\sqrt{t}}$$

$$S(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \sin t \, \frac{dt}{\sqrt{t}}$$

is given for $0 \leq x \leq 1.00$ by steps of 0.01, and for $1.0 \leq x \leq 30.0$ by steps of 0.10. In addition, values are given for $x = n\frac{\pi}{2}$, $n = 1\text{-}20$. Differences are tabulated, to facilitate interpolation. Some applications of the integrals are listed. Alternative forms and asymptotic expansions valid for large x are given.

* * * * *

The Fresnel integrals

$$C(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\cos t}{\sqrt{t}} \, dt,$$

$$S(x) = \frac{1}{\sqrt{2\pi}} \int_0^x \frac{\sin t}{\sqrt{t}} \, dt,$$

were first obtained (although not in this form) by Augustin Jean Fresnel in 1818 in the course of the development of his theory of

diffraction of light. Since then, their chief application has been in diffraction problems. However, other applications occur. In hydrodynamics, for example, the velocity potential for surface waves generated by an impulsive pressure is readily evaluated in terms of the Fresnel integrals.*

The work that initiated this tabulation was concerned with the scattering of electromagnetic radiation by cylindrical mirrors. (The Fresnel integrals arose from the contributions to the scattered field from the singularity in the surface-current density at the edge of the mirror.) When the actual evaluation of certain integrals in the formulation was begun, it was found that they could easily be cast into forms yielding the Fresnel integrals, but that the arguments were such that interpolation would frequently be required. However, it was found that no tables existed for which the tabulation was complete enough to permit satisfactory interpolation. The best previous tabulation is probably that of Watson,** which is the basis for the present work.

There are two principal forms of the integrals. The first is given above, and is used principally for convenience in tabulation. The second form is that originally used by Fresnel:

$$C(\tfrac{\pi}{2} u^2) = \int_0^u \cos(\tfrac{\pi}{2} t^2)\, dt$$

$$S(\tfrac{\pi}{2} u^2) = \int_0^u \sin(\tfrac{\pi}{2} t^2)\, dt.$$

Certain other forms also exist:

$$C(x) = \tfrac{1}{2} \int_0^x J_{-\frac{1}{2}}(t)\, dt$$

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*"Fourier Transforms," Ian N. Sneddon, McGraw-Hill, New York, 1948, pp. 278-282.

**"The Theory of Bessel Functions," G. N. Watson, Cambridge, London, 1922.

$$S(x) = \frac{1}{2}\int_0^x J_{\frac{1}{2}}(t)\,dt$$

wnen $J_{-\frac{1}{2}}$ and $J_{\frac{1}{2}}$ are Bessel functions of order $-\frac{1}{2}$ and $\frac{1}{2}$ respectively.

Further,

$$\frac{1}{2}\int_0^x H^{(2)}_{-\frac{1}{2}}(t)\,dt = C(x) - iS(x)$$

wnere $H^{(2)}_{-\frac{1}{2}}$ is the Hankel function of the second kind of order $-\frac{1}{2}$,

$$\frac{\Phi(\sqrt{ix})}{\sqrt{2i}} = C(x) - iS(x),$$

where Φ is the error function. Also

$$\int_0^u i^{-t^2}\,dt = C(\tfrac{\pi}{2}u^2) - iS(\tfrac{\pi}{2}u^2).$$

Integrals of half-order Bessel functions of order higher than $\frac{1}{2}$ may be computed by a simple integration by parts. For example,

$$\frac{1}{2}\int_0^x J_{\frac{3}{2}}(x)\,dx = C(x) - J_{\frac{1}{2}}(x).$$

This result may also be obtained from the recursion formula for the Bessel functions.

More complicated integrals may be evaluated in terms of the Fresnel integrals. For example,

$$\int_\beta^{2\pi-\beta} e^{-i\rho\cos(\alpha-\theta)}\,\frac{d\theta}{\sqrt{\theta-\beta}} = 2J_0(\rho)\sqrt{2(\pi-\beta)}$$

$$+ 2\sum_{n=1}^{\infty}(-i)^n J_n(\rho)\sqrt{2\pi n}\,\left[\cos n(\alpha+\beta)\,C\left(\tfrac{2}{n}(\pi-\beta)\right)\right.$$

$$\left. + \sin n(\alpha+\beta)\,S\left(\tfrac{2}{n}(\pi-\beta)\right)\right].$$

For $x \geq 30.0$, the following asymptotic formulas give values accurate to six decimal places:

$$C(x) \cong \frac{1}{2} + \sqrt{\frac{x}{2\pi}} \left[\sin x\left(\frac{1}{2x} - \frac{3}{8x^3}\right) - \cos x\left(\frac{1}{4x^2} - \frac{15}{16x^4}\right)\right]$$

$$S(x) \cong \frac{1}{2} - \sqrt{\frac{x}{2\pi}} \left[\cos x\left(\frac{1}{2x} - \frac{3}{8x^3}\right) + \sin x\left(\frac{1}{4x^2} - \frac{15}{16x^4}\right)\right].$$

The values of $S(x)$ at $x = n\pi$ and $C(x)$ at $x = \left(\frac{2n-1}{2}\right)\pi$ were taken from Watson.

X	S(X)	Δ	C(X)	Δ
.01	.000266	.000436	.079788	.033045
.02	.000752	.000630	.112833	.025352
.03	.001382	.000745	.138135	.021366
.04	.002127	.000846	.159551	.018817
.05	.002973	.000935	.178368	.017003
.06	.003908	.001016	.195371	.015626
.07	.004924	.001091	.210997	.014534
.08	.006015	.001162	.225531	.013641
.09	.007177	.001227	.239172	.012889
.10	.008404	.001291	.252061	.012247
.11	.009695	.001349	.264308	.011690
.12	.011044	.001407	.275998	.011198
.13	.012451	.001461	.287196	.010761
.14	.013912	.001514	.297957	.010368
.15	.015426	.001564	.308325	.010013
.16	.016990	.001614	.318338	.009689
.17	.018604	.001660	.328027	.009392
.18	.020264	.001706	.337419	.009117
.19	.021970	.001750	.346536	.008864
.20	.023720	.001794	.355400	.008628
.21	.025514	.001836	.364028	.008406
.22	.027350	.001876	.372434	.008199
.23	.029226	.001916	.380633	.008004
.24	.031142	.001955	.388637	.007819
.25	.033097	.001993	.396456	.007645
.26	.035090	.002030	.404101	.007480
.27	.037120	.002065	.411581	.007322
.28	.039185	.002101	.418903	.007172
.29	.041286	.002136	.426075	.007028
.30	.043422	.002169	.433103	.006890

X	S(X)	Δ	C(X)	Δ
.31	.045591	.002202	.439993	.006759
.32	.047793	.002235	.446752	.006632
.33	.050028	.002266	.453384	.006509
.34	.052294	.002297	.459893	.006392
.35	.054591	.002327	.466285	.006279
.36	.056918	.002357	.472564	.006168
.37	.059275	.002386	.478732	.006062
.38	.061661	.002415	.484794	.005959
.39	.064076	.002443	.490753	.005859
.40	.066519	.002470	.496612	.005762
.41	.068989	.002496	.502374	.005667
.42	.071485	.002524	.508041	.005575
.43	.074009	.002549	.513616	.005486
.44	.076558	.002574	.519102	.005398
.45	.079132	.002599	.524500	.005313
.46	.081731	.002623	.529813	.005229
.47	.084354	.002648	.535042	.005148
.48	.087002	.002670	.540190	.005067
.49	.089672	.002694	.545257	.004990
.50	.092366	.002716	.550247	.004913
.51	.095082	.002738	.555160	.004839
.52	.097820	.002760	.559999	.004764
.53	.100580	.002780	.564763	.004692
.54	.103360	.002802	.569455	.004621
.55	.106162	.002822	.574076	.004552
.56	.108984	.002841	.578628	.004487
.57	.111825	.002861	.583115	.004410
.58	.114686	.002881	.587525	.004349
.59	.117567	.002898	.591874	.004283
.60	.120465	.002918	.596157	.004219

X	S(X)	Δ	C(X)	Δ
.61	.123383	.002935	.600376	.004155
.62	.126318	.002952	.604531	.004092
.63	.129270	.002970	.608623	.004031
.64	.132240	.002986	.612654	.003969
.65	.135226	.003003	.616623	.003909
.66	.138229	.003019	.620532	.003850
.67	.141248	.003034	.624382	.003791
.68	.144282	.003050	.628173	.003733
.69	.147332	.003064	.631906	.003676
.70	.150396	.003079	.635582	.003618
.71	.153475	.003093	.639200	.003563
.72	.156568	.003107	.642763	.003507
.73	.159675	.003121	.646270	.003452
.74	.162796	.003133	.649722	.003397
.75	.165929	.003147	.653119	.003344
.76	.169076	.003159	.656463	.003291
.77	.172235	.003170	.659754	.003237
.78	.175405	.003183	.662991	.003185
.79	.178588	.003194	.666176	.003134
.80	.181782	.003205	.669310	.003082
.81	.184987	.003216	.672392	.003030
.82	.188203	.003226	.675422	.002981
.83	.191429	.003237	.678403	.002927
.84	.194666	.003246	.681330	.002884
.85	.197912	.003255	.684214	.002831
.86	.201167	.003265	.687045	.002782
.87	.204432	.003274	.689827	.002734
.88	.207706	.003281	.692561	.002686
.89	.210987	.003291	.695247	.002637
.90	.214278	.003297	.697884	.002591

X	S(X)	Δ	C(X)	Δ
.91	.217575	.003306	.700475	.002543
.92	.220881	.003312	.703018	.002496
.93	.224193	.003320	.705514	.002450
.94	.227513	.003326	.707964	.002404
.95	.230839	.003333	.710368	.002358
.96	.234172	.003338	.712726	.002313
.97	.237510	.003344	.715039	.002267
.98	.240854	.003350	.717306	.002222
.99	.244204	.003354	.719528	.002178
1.00	.247558		.721706	

X	S(X)	Δ	C(X)	Δ
1.0	.247558		.721706	
		+.033759		.019383
1.1	.281317	.033945	.741089	.015206
1.2	.315262	.033851	.756295	.011260
1.3	.349113	.033491	.767555	.007529
1.4	.382604	.032879	.775084	.004000
1.5	.415483	.032017	.779084	+.000679
1.6	.447500	.030962	.779763	-.002453
1.7	.478462	.029656	.777310	-.005362
1.8	.508118	.028198	.771948	-.008079
1.9	.536316	.026533	.763869	-.010567
2.0	.562849	.024720	.753302	-.012837
2.1	.587569	.022775	.740465	-.014888
2.2	.610344	.020679	.725577	-.016692
2.3	.631023	.018523	.708885	-.018283
2.4	.649546	.016241	.690602	-.019616
2.5	.665787	.013930	.670986	-.020728
2.6	.679717	.011568	.650258	-.021594
2.7	.691285	.009181	.628664	-.022227
2.8	.700466	.006795	.606437	-.022624
2.9	.707261	.004424	.585813	-.022793
3.0	.711685	+.002091	.561020	-.022738
3.1	.713776	-.000185	.538282	-.022469
3.2	.713591	-.002391	.515813	-.021991
3.3	.711200	-.004505	.493822	-.021316
3.4	.706695	-.006515	.472506	-.020459
3.5	.700180	-.008403	.452047	-.019425
3.6	.691777	-.010159	.432622	-.018235
3.7	.681618	-.011768	.414387	-.016899
3.8	.669850	-.013222	.397488	-.015436
3.9	.656628	-.014509	.382052	-.013859

X	S(X)	Δ	C(X)	Δ
4.0	.642119		.368193	
		-.014625		-.012189
4.1	.626494		.356004	
		-.016558		-.010439
4.2	.609936		.345565	
		-.017313		-.008631
4.3	.592623		.336934	
		-.017876		-.006780
4.4	.574747		.330154	
		-.018258		-.004905
4.5	.556489		.325249	
		-.018448		-.004415
4.6	.538041		.320834	
		-.018458		+.000236
4.7	.519583		.321070	
		-.018284		.000687
4.8	.501299		.321757	
		-.017937		.002482
4.9	.483362		.324239	
		-.017420		.004218
5.0	.465942		.328457	
		-.016744		.005876
5.1	.449198		.334333	
		-.015917		.007446
5.2	.433281		.341779	
		-.014950		.008911
5.3	.418331		.350690	
		-.013854		.010262
5.4	.404477		.360952	
		-.012643		.011487
5.5	.391834		.372439	
		-.011330		.012577
5.6	.380504		.385016	
		-.009927		.013524
5.7	.370577		.398540	
		-.008455		.014321
5.8	.362122		.412861	
		-.006921		.014965
5.9	.355201		.427826	
		-.005349		.015448
6.0	.349852		.443274	
		-.003748		.015774
6.1	.346104		.459048	
		-.002136		.015937
6.2	.343968		.474985	
		-.000531		.015943
6.3	.343437		.490928	
		+.001056		.015789
6.4	.344493		.506717	
		.002606		.015485
6.5	.347099		.522202	
		.004109		.015030
6.6	.351208		.537232	
		.005543		.014436
6.7	.356751		.551668	
		.006909		.013707
6.8	.363660		.565375	
		.008178		.012855
6.9	.371838		.578230	
		.009357		.011886

X	S(X)	Δ	C(X)	Δ
7.0	.381195	.010419	.590116	.010816
7.1	.391614	.011369	.600932	.009654
7.2	.402983	.012188	.610586	.008411
7.3	.415171	.012882	.618997	.007105
7.4	.428053	.013432	.626102	.005743
7.5	.441485	.013848	.631845	.004345
7.6	.455333	.014118	.636190	.002921
7.7	.469451	.014248	.639111	+.001488
7.8	.483699	.014232	.640599	+.000057
7.9	.497931	.014079	.640656	-.001355
8.0	.512010	.013785	.639301	-.002736
8.1	.525795	.013362	.636565	-.004075
8.2	.539157	.012809	.632490	-.005354
8.3	.551966	.012138	.627136	-.006568
8.4	.564104	.011353	.620568	-.007709
8.5	.575457	.010467	.612868	-.008743
8.6	.585824	.009485	.604125	-.009689
8.7	.595409	.008423	.594436	-.010526
8.8	.603832	.007288	.583910	-.011251
8.9	.611120	.006094	.572659	-.011855
9.0	.617214	.004853	.560804	-.012336
9.1	.622067	.003578	.548468	-.012688
9.2	.625645	.002280	.535780	-.012912
9.3	.627925	+.000975	.522868	-.013005
9.4	.628900	-.000327	.509863	-.012968
9.5	.628573	-.001612	.496895	-.012803
9.6	.626961	-.002865	.484092	-.012513
9.7	.624096	-.004081	.471579	-.012102
9.8	.620015	-.005240	.459477	-.011575
9.9	.614775	-.006339	.447902	-.010938

X	S(X)	Δ	C(X)	Δ
10.0	.608436		.436964	
		-.007362		-.010199
10.1	.601074		.426765	
		-.008302		-.009368
10.2	.592772		.417397	
		-.009151		-.008450
10.3	.583621		.408947	
		-.009900		-.007458
10.4	.573721		.401489	
		-.010545		-.006402
10.5	.563176		.395087	
		-.011080		-.005294
10.6	.552096		.389793	
		-.011495		-.004137
10.7	.540601		.385656	
		-.011800		-.002960
10.8	.528801		.382696	
		-.011976		-.001754
10.9	.516825		.380942	
		-.012041		-.000552
11.0	.504784		.380390	
		-.011978		+.000651
11.1	.492806		.381041	
		-.011801		.001838
11.2	.481005		.382879	
		-.011506		.002990
11.3	.469499		.385869	
		-.011102		.004110
11.4	.458397		.389979	
		-.010588		.005170
11.5	.447809		.395149	
		-.009976		.006180
11.6	.437833		.401329	
		-.009270		.007108
11.7	.428563		.408437	
		-.008477		.007971
11.8	.420086		.416408	
		-.007607		.008733
11.9	.412479		.425141	
		-.006669		.009416
12.0	.405810		.434557	
		-.006671		.009990
12.1	.400139		.444547	
		-.005629		.010463
12.2	.395510		.455010	
		-.003544		.010828
12.3	.391966		.465838	
		-.002438		.011083
12.4	.389528		.476921	
		-.001311		.011225
12.5	.388217		.488146	
		-.000184		.011256
12.6	.388033		.499402	
		+.000936		.011172
12.7	.388969		.510574	
		.002039		.010981
12.8	.391008		.521555	
		.003113		.010679
12.9	.394121		.532234	
		.004147		.010277

X	S(X)	Δ	C(X)	Δ
13.0	.398268		.542511	
		.005133		.009773
13.1	.403401		.552284	
		.006059		.009177
13.2	.409460		.561461	
		.006920		.008493
13.3	.416380		.569954	
		.007703		.007732
13.4	.424083		.577686	
		.008406		.006897
13.5	.432489		.584583	
		.009018		.006002
13.6	.441507		.590585	
		.009538		.005053
13.7	.451045		.595638	
		.009958		.004061
13.8	.461003		.599699	
		.010277		.003035
13.9	.471280		.602734	
		.010490		.001987
14.0	.481770		.604721	
		.010598		+.006057
14.1	.492368		.610778	
		.010601		-.005266
14.2	.502969		.605512	
		.010496		-.001190
14.3	.513465		.604322	
		.010290		-.002222
14.4	.523755		.602100	
		.009981		-.003229
14.5	.533736		.598871	
		.009576		-.004195
14.6	.543312		.594676	
		.009079		-.005111
14.7	.552391		.589565	
		.008493		-.005973
14.8	.560884		.583592	
		.007830		-.006766
14.9	.568714		.576826	
		.007089		-.007491
15.0	.575803		.569335	
		.006285		-.008132
15.1	.582088		.561203	
		.005426		-.008692
15.2	.587514		.552511	
		.004514		-.009159
15.3	.592028		.543352	
		.003567		-.009533
15.4	.595595		.533819	
		.002588		-.009810
15.5	.598183		.524009	
		.001592		-.009986
15.6	.599775		.514023	
		+.000583		-.010064
15.7	.600358		.503959	
		-.000421		-.010039
15.8	.599937		.493920	
		-.001419		-.009916
15.9	.598518		.484004	
		-.002392		-.009694

X	S(X)	Δ	C(X)	Δ
16.0	.596126		.474310	
		-.003338		-.009377
16.1	.592788		.464933	
		-.004245		-.008970
16.2	.588543		.455963	
		-.005103		-.008474
16.3	.583440		.447489	
		-.005906		-.007899
16.4	.577534		.439590	
		-.006644		-.007247
16.5	.570890		.432343	
		-.007312		-.006527
16.6	.563578		.425816	
		-.007904		-.005749
16.7	.555674		.420067	
		-.008413		-.004915
16.8	.547261		.415152	
		-.008835		-.004040
16.9	.538426		.411112	
		-.009167		-.003127
17.0	.529259		.407985	
		-.009406		-.002011
17.1	.519853		.405794	
		-.009550		-.001236
17.2	.510303		.404558	
		-.009597		-.000277
17.3	.500706		.404281	
		-.009550		+.000682
17.4	.491156		.404963	
		-.009406		.001626
17.5	.481750		.406589	
		-.009171		.002551
17.6	.472579		.409140	
		-.008846		.003442
17.7	.463733		.412582	
		-.008433		.004298
17.8	.455300		.416880	
		-.007941		.005102
17.9	.447359		.421982	
		-.007370		.005855
18.0	.439989		.427837	
		-.006730		.006542
18.1	.433259		.434379	
		-.006028		.007162
18.2	.427231		.441541	
		-.005267		.007706
18.3	.421964		.449247	
		-.004460		.008172
18.4	.417504		.457419	
		-.003611		.008552
18.5	.413893		.465971	
		-.002733		.008848
18.6	.411160		.474819	
		-.001830		.009049
18.7	.409330		.483868	
		-.000916		.009165
18.8	.408414		.493033	
		+.000004		.009184
18.9	.408418		.502217	
		.000918		.009115

X	S(X)	Δ	C(X)	Δ
19.0	.409336		.511332	
		.001819		.008952
19.1	.411155		.520284	
		.002697		.008706
19.2	.413852		.528990	
		.003543		.008369
19.3	.417395		.537359	
		.004350		.007955
19.4	.421745		.545314	
		.005108		.007460
19.5	.426853		.552774	
		.005814		.006896
19.6	.432667		.559670	
		.006455		.006265
19.7	.439122		.565935	
		.007032		.005575
19.8	.446154		.571510	
		.007533		.004833
19.9	.453687		.576343	
		.007959		.004046
20.0	.461646		.580389	
		.008303		.003225
20.1	.469949		.583614	
		.008560		.002371
20.2	.478509		.585985	
		.008734		.001504
20.3	.487243		.587489	
		.008817		+.000620
20.4	.496060		.588109	
		.008815		-.000260
20.5	.504875		.587849	
		.008666		-.001137
20.6	.513541		.586712	
		.008617		-.001998
20.7	.522158		.584714	
		.008269		-.002833
20.8	.530427		.581881	
		.007968		-.003638
20.9	.538395		.578243	
		.007490		-.004401
21.0	.545885		.573842	
		.007024		-.005118
21.1	.552909		.568724	
		.006465		-.005781
21.2	.559374		.562943	
		.005839		-.006381
21.3	.565213		.556562	
		.005164		-.006917
21.4	.570377		.549645	
		.004434		-.007379
21.5	.574811		.542266	
		.003667		-.007768
21.6	.578478		.534498	
		.002869		-.008076
21.7	.581347		.526422	
		.002041		-.008304
21.8	.583388		.518118	
		.001202		-.008445
21.9	.584590		.509673	
		+.000349		-.008506

X	S(X)	Δ	C(X)	Δ
22.0	.584939		.501167	
		-.000499		-.008478
22.1	.584440		.492689	
		-.001341		-.008366
22.2	.583099		.484323	
		-.002163		-.008172
22.3	.580936		.476151	
		-.002963		-.007898
22.4	.577973		.468253	
		-.003727		-.007546
22.5	.574246		.460707	
		-.004453		-.007321
22.6	.569793		.453386	
		-.005129		-.006425
22.7	.564664		.446961	
		-.005753		-.006067
22.8	.558911		.440894	
		-.006316		-.005450
22.9	.552595		.435444	
		-.006813		-.004782
23.0	.545782		.430662	
		-.007241		-.004069
23.1	.538541		.426593	
		-.007595		-.003319
23.2	.530946		.423274	
		-.007871		-.002538
23.3	.523075		.420736	
		-.008068		-.001737
23.4	.515007		.418999	
		-.008183		-.000919
23.5	.506824		.418080	
		-.008216		-.000099
23.6	.498608		.417981	
		-.008169		+.000722
23.7	.490439		.418703	
		-.008038		.001529
23.8	.482401		.420232	
		-.007830		.002320
23.9	.474571		.422552	
		-.007542		.003083
24.0	.467029		.425635	
		-.007183		.003813
24.1	.459846		.429448	
		-.006751		.004501
24.2	.453095		.433949	
		-.006256		.005143
24.3	.446839		.439092	
		-.005699		.005729
24.4	.441140		.444821	
		-.005089		.006257
24.5	.436051		.451078	
		-.004429		.006720
24.6	.431622		.457798	
		-.003729		.007114
24.7	.427893		.464912	
		-.002993		.007436
24.8	.424900		.472348	
		-.002233		.007682
24.9	.422667		.480030	
		-.001450		.007850

X	S(X)	Δ	C(X)	Δ
25.0	.421217		.487880	
		-.000658		.007940
25.1	.420559		.495820	
		+.000036		.007951
25.2	.420695		.503771	
		.000929		.007881
25.3	.421624		.511652	
		.001706		.007735
25.4	.423330		.519387	
		.002467		.007509
25.5	.425797		.526896	
		.003197		.007209
25.6	.428994		.534105	
		.003894		.006863
25.7	.432888		.540968	
		.004548		.006393
25.8	.437436		.547361	
		.005156		.005926
25.9	.442592		.553287	
		.005708		.005341
26.0	.448300		.558628	
		.006203		.004751
26.1	.454503		.563379	
		.006634		.004099
26.2	.461137		.567478	
		.006996		.003411
26.3	.468133		.570889	
		.007288		.002689
26.4	.475421		.573578	
		.007506		.001946
26.5	.482927		.575524	
		.007648		.001183
26.6	.490575		.576707	
		.007714		+.000414
26.7	.498289		.577121	
		.007702		-.000358
26.8	.505991		.576763	
		.007613		-.001123
26.9	.513604		.575640	
		.007450		-.001874
27.0	.521054		.573766	
		.007211		-.002604
27.1	.528265		.571162	
		.006904		-.003304
27.2	.535169		.567858	
		.006526		-.003970
27.3	.541695		.563888	
		.006087		-.004593
27.4	.547782		.559295	
		.005587		-.005168
27.5	.553369		.554127	
		.005035		-.005690
27.6	.558404		.548437	
		.004433		-.006153
27.7	.562837		.542284	
		.003790		-.006553
27.8	.566627		.535731	
		.003111		-.006886
27.9	.569738		.528845	
		.002404		-.007150

X	S(X)	Δ	C(X)	Δ
28.0	.572142		.521695	
		.001676		-.007341
28.1	.573818		.514354	
		.000931		-.007458
28.2	.574749		.506896	
		+.000184		-.007501
28.3	.574933		.499395	
		-.000567		-.007468
28.4	.574366		.491927	
		-.001306		-.007361
28.5	.573060		.484566	
		-.002032		-.007181
28.6	.571028		.477385	
		-.002733		-.006931
28.7	.568295		.470454	
		-.003406		-.006611
28.8	.564889		.463843	
		-.004041		-.006228
28.9	.560848		.457615	
		-.004636		-.005783
29.0	.556212		.451832	
		-.005180		-.005284
29.1	.551032		.446548	
		-.005673		-.004729
29.2	.545359		.441819	
		-.006105		-.004135
29.3	.539254		.437684	
		-.006477		-.003496
29.4	.532777		.434188	
		-.006782		-.002830
29.5	.525995		.431358	
		-.007017		-.002131
29.6	.518978		.429227	
		-.007186		-.001423
29.7	.511792		.427804	
		-.007277		-.000693
29.8	.504515		.427111	
		-.007300		+.000034
29.9	.497215		.427145	
		-.007246		.000763
30.0	.489969		.427908	

Values of $C(x)$ at $x = (2n-1)\frac{\pi}{2}$

N	C(x)
1	0.779893
2	0.321056
3	0.640807
4	0.380389
5	0.605721
6	0.404260
7	0.588128
8	0.417922
9	0.577121
10	0.427036

Values of $S(x)$ at $x = n\pi$

N	S(x)
1	0.713972
2	0.343415
3	0.628940
4	0.387969
5	0.600361
6	0.408301
7	0.584942
8	0.420516
9	0.574957
10	0.428877